Integrals of the Full Symmetric Toda System

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Content

Toda System

Noneperiodic Toda system – the system of n particles on a line with the interactions between neighborhoods.

$$H = \sum_{i=1}^{n} \frac{1}{2} p_i^2 + \sum_{i=1}^{n-1} \exp(q_i - q_{i+1}), \tag{1}$$

where p_i – momentum of the particle, a q_i – its coordinate. Poisson structure

$$\{p_i, q_j\} = \delta_{ij},\ \{p_i, p_j\} = 0,\ \{q_i, q_j\} = 0. \tag{2}$$

Flaschka's variables

$$b_i = p_i, \quad a_i = \exp \frac{1}{2}(q_i - q_{i+1}), \tag{3}$$

$$H = \sum_{i=1}^{n} \frac{1}{2} b_i^2 + \sum_{i=1}^{n-1} a_i^2. \tag{4}$$

$$\{b_i, a_{i-1}\} = -a_{i-1},\ \{b_i, a_i\} = a_i. \tag{5}$$

Lax representation

Toda system has the Lax representation

$$L^{'} = [B, L], \tag{6}$$

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & a_{n-2} & b_{n-1} & a_{n-1} \\ 0 & 0 & \dots & a_{n-1} & b_n \end{pmatrix} \tag{7}$$

$$B = \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 \\ -a_1 & 0 & a_{23} & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & -a_{n-2} & 0 & a_{n-1} \\ 0 & 0 & \dots & -a_{n-1} & 0 \end{pmatrix} \equiv L_{>0} - L_{<0} \tag{8}$$

(6) is condition of the compatibility of the system

$$\begin{cases} L\Psi = \Psi\Lambda, \\ \frac{\partial}{\partial t}\Psi = B\Psi, \end{cases} \tag{9}$$

where $\Psi \in SO(n, \mathbb{R})$, L is symmetric matrix, Λ is diagonal matrix. Dimension of the phase space is $2(n-1)$. This system is integrable with the integrals of motion

$$H_k = \frac{1}{k} TrL^k, \; k = \overline{1, n}. \tag{10}$$

Generalization. Full Symmetric Toda System

$$L = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix},$$

$$B = (\psi \Lambda \psi^T)_{>0} - (\psi \Lambda \psi^T)_{<0} = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & \dots & a_{1n} \\ -a_{12} & 0 & a_{23} & \dots & \dots & a_{2n} \\ -a_{13} & -a_{23} & 0 & \dots & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & a_{n-1n} \\ -a_{1n} & -a_{2n} & -a_{3n} & \dots & -a_{n-1n} & 0 \end{pmatrix}. \tag{11}$$

This system – named Full Symmetric Toda System – is integrable too. But the number of the isospectral integrals of motion is not sufficient for the integrability.

Integrability. Chopping procedure

P. Deift, L. C. Li, T. Nanda, and C. Tomei, The Toda flow on a generic orbit is integrable, (1986)

$$L = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \tag{12}$$

Define the set of the characteristic polynomials

$$P_k(L,\mu) = det(L - \mu I)_k,$$
$$P_k(L,\mu) = \sum_{m=0}^{n-2k} E_{m,k}(L)\mu^{n-2k-m}, \;\; 0 \leqslant k \leqslant [n/2], \tag{13}$$

matrix $(L - \mu I)_k$ of the order $n - k$, which is obtained by the cutting the number of k upper raws and k right columns of the matrix $(L - \mu I)$, [ ] – integer part. The functions

$$I_{m,k} = \frac{E_{m,k}(L)}{E_{0,k}(L)}, \quad 0 \leqslant k \leqslant [\frac{1}{2}(n-1)], \quad 1 \leqslant m \leqslant n - 2k \tag{14}$$

define $[\frac{1}{4}n^2]$ integrals in involution on a generic orbit of dimension $2[\frac{1}{4}n^2]$. These integrals are functionally independent.

New way to construct the integrals

$$J_{k_1,k_2} = \frac{A^{(k_1)}_{\frac{n-m+1,\ldots,n}{1,2,\ldots,m}}}{A^{(k_2)}_{\frac{n-m+1,\ldots,n}{1,2,\ldots,m}}}. \tag{15}$$

Minor $A^{(k_1)}_{\frac{n-m+1,\ldots,n}{1,2,\ldots,m}}$ is the left lower angle minor of $L^{k_i} = \underbrace{L \cdot L \cdot \ldots \cdot L}_{k_i}$

Example. N=4

$$L = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

$$L^{'} = [B, L].$$

$$L = \Psi \Lambda \Psi^{-1}, \quad a_{ij} = \sum_{k=1}^{4} \lambda_k \psi_{ik} \psi_{jk}.$$

$$L\Psi = \Psi\Lambda, \; \Rightarrow \; (L^k \Psi = \Psi \Lambda^k) \Rightarrow a_{ij}^k = \sum_{l=1}^{4} \lambda_l^k \psi_{il} \psi_{jl}$$

$$(a_{14}^{(k)})' = (a_{44} - a_{11}) a_{14}^{(k)}, \quad a_{14}^{(k)} = \lambda_1^k \psi_{11}\psi_{41} + \lambda_2^k \psi_{12}\psi_{42} + \lambda_3^k \psi_{13}\psi_{43} + \lambda_4^k \psi_{14}\psi_{44}.$$

$$(A^{(k)}_{\frac{34}{12}})' = (-2(a_{11} + a_{22}) + TrL) A^k_{\frac{34}{12}},$$

$$A^{(k)}_{\frac{34}{12}} = (\lambda_1^k \lambda_2^k + \lambda_3^k \lambda_4^k) M_{\frac{12}{34}} M_{\frac{12}{12}} + (-\lambda_1^k \lambda_3^k - \lambda_2^k \lambda_4^k) M_{\frac{12}{13}} M_{\frac{12}{24}} + (\lambda_1^k \lambda_4^k + \lambda_2^k \lambda_3^k) M_{\frac{12}{23}} M_{\frac{12}{14}}.$$

Example n=4. Integrals

The dynamics of Ψ

$$\begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} \\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_{44} \end{pmatrix}' =$$

$$\begin{pmatrix} (-a_{11}+\lambda_1)\psi_{11} & (-a_{11}+\lambda_2)\psi_{12} & (-a_{11}+\lambda_3)\psi_{13} & (-a_{11}+\lambda_4)\psi_{14} \\ \begin{matrix}(-a_{22}+\lambda_1)\psi_{21}- \\ -2a_{12}\psi_{11}\end{matrix} & \begin{matrix}(-a_{22}+\lambda_2)\psi_{22}- \\ -2a_{12}\psi_{12}\end{matrix} & \begin{matrix}(-a_{22}+\lambda_3)\psi_{23}- \\ -2a_{12}\psi_{13}\end{matrix} & \begin{matrix}(-a_{22}+\lambda_3)\psi_{24}- \\ -2a_{12}\psi_{14}\end{matrix} \\ \begin{matrix}(-a_{33}+\lambda_1)\psi_{31}- \\ -2a_{13}\psi_{11}-2a_{23}\psi_{21}\end{matrix} & \begin{matrix}(-a_{33}+\lambda_2)\psi_{32}- \\ -2a_{13}\psi_{12}-2a_{23}\psi_{22}\end{matrix} & \begin{matrix}(-a_{33}+\lambda_3)\psi_{33}- \\ -2a_{13}\psi_{13}-2a_{23}\psi_{23}\end{matrix} & \begin{matrix}(-a_{33}+\lambda_4)\psi_{34}- \\ -2a_{13}\psi_{14}-2a_{23}\psi_{24}\end{matrix} \\ (a_{44}-\lambda_1)\psi_{41} & (a_{44}-\lambda_2)\psi_{42} & (a_{44}-\lambda_3)\psi_{43} & (a_{44}-\lambda_4)\psi_{44} \end{pmatrix}$$

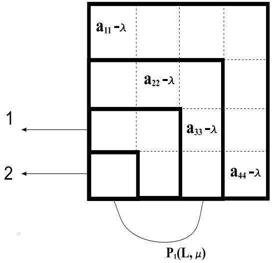
$$TrL, \quad \frac{1}{2}TrL^2, \quad \frac{1}{3}TrL^3, \quad \frac{1}{4}TrL^4, \quad I_{1,1} = \frac{a_{14}^{(2)}}{a_{14}}, \quad I_{2,1} = \frac{a_{14}^{(3)}}{a_{14}}$$

The dimension of the phase space is 8. But there exists additional integral

$$J = \frac{A^{(2)}_{\frac{34}{12}}}{A_{\frac{34}{12}}},$$

which commutates with $\frac{1}{k}TrL^k$ and do not commutates with I.

n=4



Main Theorems
In general case

Theorem 1

In matrix $\Psi \in SO(n, \mathbb{R})$ the intersections of the first upper $k < n$ rows or first lower $l < n$ rows with any set of k or l columns accordingly are semi-invariants with regard to the actions of 1-parametric subgroups induced by $\frac{1}{d} TrL^d$, $d \in \mathbb{N}$. Equations of motion $(M, \tilde{M})$:

$$\frac{\partial}{\partial t_{d-1}} M_{\frac{1,2,\ldots,k}{i_1,i_2,\ldots,i_k}} = \left(-\sum_{j=1}^{k} a_{jj}^{(d-1)} + \sum_{i_m=i_1}^{i_k} \lambda_{i_m}^{d-1}\right) M_{\frac{1,2,\ldots,k}{i_1,i_2,\ldots,i_k}},$$

$$\frac{\partial}{\partial t_{d-1}} \tilde{M}_{\frac{n-l+1,\ldots,n}{i_1,i_2,\ldots,i_l}} = \left(\sum_{j=n-l}^{n} a_{jj}^{(d-1)} - \sum_{i_m=i_1}^{i_l} \lambda_{i_m}^{d-1}\right) \tilde{M}_{\frac{n-l+1,\ldots,n}{i_1,i_2,\ldots,i_l}}.$$

Theorem 2

Minors of L^k $A^{(k)}_{\frac{n-m+1,\ldots,n}{1,2,\ldots,m}}$, $n > m$ are semi-invariants with regard to the actions of 1-parametric subgroups induced by $\frac{1}{d} TrL^d$, $d \in \mathbb{N}$.

$$A^{(k)}_{\frac{n-m+1,\ldots,n}{1,2,\ldots,m}} = \sum_{i_1,i_2,\ldots,i_m} \lambda_{i_1}^k \lambda_{i_2}^k \cdot \ldots \cdot \lambda_{i_m}^k M_{\frac{1,2,\ldots,m}{i_1,i_2,\ldots,i_m}} M_{\frac{n-m+1,\ldots,n}{i_1,i_2,\ldots,i_m}}.$$

$$J_{k_1,k_2} = \frac{A^{(k_1)}_{\frac{n-m+1,\ldots,n}{1,2,\ldots,m}}}{A^{(k_2)}_{\frac{n-m+1,\ldots,n}{1,2,\ldots,m}}}.$$

Plukker's coordinates. Number of integrals

$$\mathbb{FL}_n(\mathbb{R}) \hookrightarrow \mathbb{RP}^{n-1} \times \ldots \times \mathbb{RP}^{C^n_{k_1}-1} \times \ldots \times (\mathbb{RP}^{C^n_{k_2}-1})^* \times \ldots \times (\mathbb{RP}^{n-1})^*,$$

$$1 \leqslant k_1 \leqslant [\frac{n}{2}] < k_2 \leqslant n-1.$$

On vector space $V^n = \mathbb{R}^n$ - basis $\{e_i\}$ and Plukker's coordinates

$$X_{i_1,i_2,\ldots,i_m} = M_{\frac{1,2,\ldots,m}{i_1,i_2,\ldots,i_m}}(\psi).$$

Invariants

$$\varphi(M(\psi)) = \frac{M_{\frac{1,2,\ldots,m}{i_1,i_2,\ldots,i_m}} M_{\frac{n-m+1,\ldots,n}{i_1,i_2,\ldots,i_m}}}{M_{\frac{1,2,\ldots,m}{j_1,j_2,\ldots,j_m}} M_{\frac{n-m+1,\ldots,n}{j_1,j_2,\ldots,j_m}}},$$

Theorem 3

The number of the functionally independent integrals N_ψ constructed by $M(\psi)$ is equal to

$$dimFl_n(\mathbb{R}) - (n-1).$$

The full number of integrals in non-commutative family

$$N_n = \frac{1}{2}n(n-1) - [\frac{1}{2}(n+1)] + 1. \quad (16)$$

Involution families

In the case $n = 4$ there are two involution families. Each of these families makes the full Toda system integrable:
– Iso-spectral Integrals $H_k = \frac{1}{k} TrL^k$ and Integrals obtained from chopping procedure,
– Iso-spectral Integrals $H_k = \frac{1}{k} TrL^k$ and Additional Integrals.
In general case it follows from the analysis of the formula for full number of integrals

$$N_n = \frac{1}{2}n(n-1) - [\frac{1}{2}(n+1)] + 1,$$

$$N_n = N_n^{Chev} + N_n^{Chopp} + N_n^{Add},$$

that

$$N_n^{Chopp} = N_n^{Add}$$

So, there are no more than two families in involution which are possible to extract from any full non-commutative set of integrals of the Full Symmetric Toda System. It does not eliminate the existing of many families in involution from different sets of integrals.

Example n=5

$$N_n = \frac{1}{2}5(5-1) - [\frac{1}{2}(5+1)] + 1 = 8.$$

$$\frac{1}{2}TrL^2, \ \frac{1}{3}TrL^3, \ \frac{1}{4}TrL^4, \ \frac{1}{5}TrL^5, \ I_{2,1} = \frac{A^3_{51}}{a_{15}}, \ I_{3,1} = \frac{A^4_{51}}{a_{15}}.$$

$$\frac{1}{2}TrL^2, \ \frac{1}{3}TrL^3, \ \frac{1}{4}TrL^4, \ \frac{1}{5}TrL^5, \ J_1 = \frac{A^{(2)}_{\frac{45}{12}}}{A_{\frac{45}{12}}}, \ J_2 = \frac{A^{(3)}_{\frac{45}{12}}}{A_{\frac{45}{12}}}.$$

n=5